Twelve hours after a large quantity of paler urine was passed, in which not a trace of either salt injected was discovered.

Forty-eight hours after the dog had been first injected it was killed.

There were about two ounces of bloody serum in the abdomen.

Not a trace of iron or potash were found in this serum or in any of the tissues.

No prussian blue in the thoracic duct or elsewhere.

Experiment 4.—Injected one and a-half ounce of warm saturated solution of ferrocyanide of potassium beneath the skin of the back of a large dog. Some hours after urine was voided, containing abundance of the salt injected. This upon evaporation yielded yellow crystals. No trace of blue.

Experiment 5.—Saturated some human urine with ferrocyanide of potassium. On evaporation crystals, as in the last experiment. No blue colour.

It thus appears that when iron is introduced into the system, it passes off by the kidneys in combination with some other constituent, from which it is only separated by heat, when, if the ferrocyanide of potassium is present, prussian blue is immediately formed. It does not appear that the ferrocyanide of potassium is in any other state than solution. Neither of the salts were ever detected in the fæces; the same is true of the aniline dyes.

It is known that ferrocyanide of potassium, by exposure, assumes a blue tint, yet is still crystalline. Experiments 4 and 5 were instituted to see if, by the process of evaporation, a similar colour could be produced, so as to cause a doubtful result. They prove clearly the presence in Experiments 1, 2, and 3, of a far larger amount of iron than is contained in any ferrocyanide, sufficient indeed to produce the well-known amorphous prussian blue of commerce.

Whilst these experiments show no such combination takes place within the body, still they reveal the presence of a salt in the blood, chyle, and urine, which upon evaporation at ordinary temperatures appears as minute bright blue crystals.

ART. VI.—*On Differential Equations and on Co-resolvents.* By the HONOURABLE CHIEF JUSTICE COCKLE, F.R.S., President of the Queensland Philosophical Society, &c. Communicated by the HONOURABLE SIR REDMOND BARRY, Chancellor of the University of Melbourne, &c.

[Read 11th June, 1866.]

§ 1. *On Differential Equations.*

1. I propose to show that any linear differential equation whatever can be deprived of its second and third terms simultaneously, provided that we are at liberty to assume the solution of the general linear differential of the second order; or, in other words, that the annihilation of the second and third terms of any linear differential equation may be made to depend upon the solution of a linear differential equation of the second order. This is the analogue of the proposition that the second and third terms of an algebraical equation may be made to vanish simultaneously by means of the solution of a quadratic only.

2. Since any linear differential equation can be deprived of its second term by solving a linear differential equation of the first order only, we are at liberty to start from the equation of the third order—

$$\frac{d^3 y}{d x^3} + 3 r \frac{d y}{d x} + s y = 0 \qquad - \quad - (a)$$

3. Now, inasmuch as the complexity of the formulæ renders it necessary, or at all events desirable, to abridge the notation as much as possible, I shall have recourse to the following abbreviations. I shall denote differentiations with respect to the variable x by acute accents ($'$), and differentiations with respect to a new independent variable t by grave accents ($\grave{}$). Thus I shall write—

$$\frac{d y}{d x} = y', \frac{d^2 y}{d x^2} = y'', \frac{d^3 y}{d x^3} = y''', \text{ \&c.}$$

$$\frac{d y}{d t} = y^{\backprime}, \frac{d^2 y}{d t^2} = y^{\backprime\backprime}, \frac{d^2 y}{d t^2} = y^{\backprime\backprime\backprime}, \text{ \&c.}$$

The letters r and s in equation (a) denote any constants or any functions of x, the multiplier '3' being prefixed to r

for convenience only. In the abridged notation (a) will be written

$$y''' + 3\,r\,y' + s\,y = 0 \quad - \quad - \quad - \quad (b)$$

4. Change the dependent variable from y to Y where y and Y are connected by the relation

$$y = u\,Y \quad - \quad - \quad - \quad - \quad (c)$$

proceeding as follows :—

$$\begin{aligned} y''' &= \quad (u\,Y)''' = u\,Y''' + 3\,u'\,Y'' + 3\,u''\,Y' + u'''\,Y, \\ 3\,r\,y' &= 3\,r\,(u\,Y)' = \qquad\qquad\qquad\qquad 3\,r\,u\,Y' + 3\,r\,u'\,Y \\ s\,y &= \quad s\,(u\,Y) = \qquad\qquad\qquad\qquad\qquad\qquad s\,u\,Y \end{aligned}$$

whence, bearing in mind the relation (b),

$$u\,Y''' + 3\,u'\,Y'' + 3\,(u'' + r\,u)\,Y' + (u''' + 3\,r\,u' + s\,u)\,Y = 0 \ - \ (d)$$

or, dividing by u,

$$Y''' + 3\frac{u'}{u}\,Y'' + 3\left(\frac{u''}{u} + r\right)Y' + \left(\frac{u'''}{u} + 3\,r\,\frac{u'}{u} + s\right)Y = 0 \ - \ (e)$$

5. Next change the independent variable from x to t and we find, after dividing by $\left(\frac{d\,t}{d\,x}\right)^2$ or $(t')^2$, that (e) becomes, if we confine, for a moment, the change to Y', Y'', and Y'''

$$\left.\begin{aligned} Y''' + 3\left(\frac{u'}{u}\cdot x' - \frac{x''}{x'}\right)Y'' + \left\{ 3\left(\frac{u''}{u} + r\right)(x')^2 - 3\,\frac{u'}{u}\,x'' \right. \\ \left. + 3\left(\frac{x''}{x'}\right)^2 - \frac{x'''}{x'} \right\} Y' + F\,(u, x, t).\ Y = 0 \end{aligned}\right\} (f)$$

where $F\,(u, x, t)$ is a function which it is not at present necessary to develope.

6. Hence, one of the conditions of the proposed transformation being that the second term of (f) should disappear, or, what amounts to the same thing, that the co-efficient of Y'' should vanish, we have, first,

$$\frac{u'}{u}\,.\,x' - \frac{x''}{x'} = 0 \quad - \quad - \quad - \quad - \quad (g)$$

which reduces to

$$\frac{u'}{u} - \frac{x''}{x'} = 0 \quad - \quad - \quad - \quad - \quad - \quad (h)$$

of which equation a first integral is

$$u = C\,x' \qquad (i)$$

whence follow

$$u' = C\,x'', \; u'' = C\,x''', \text{ \&c.} \qquad (j, k \,.\,.)$$

7. Our next object is to eliminate u from the co-efficient of Y', and for this purpose we deduce at once from (g) the relation

$$\frac{u'}{u} = \frac{x''}{(x')^2} \qquad (l)$$

moreover

$$\frac{u''}{u} = \left\{ \frac{u''}{(x')^2} - \frac{x''}{(x')^3} \cdot u' \right\} \frac{1}{u} \qquad (m)$$

$$= \frac{x'''}{(x')^3} - \frac{(x'')^2}{(x')^4} \qquad (n)$$

as we see from inspecting (i) and $(j, k, \,.\,.)$. Hence the co-officient of Y' in (f) becomes, on substitution,

$$3\left\{ \frac{x'''}{(x')^3} - \frac{(x'')^2}{(x')^4} + r \right\} (x')^2 - 3\,\frac{(x'')^2}{(x')^2} + 3\left(\frac{x''}{x'}\right)^2 - \frac{x'''}{x'}$$

and reducing this expression and equating the result to zero we find, as the second condition of the proposed transformation,

$$2\,\frac{x'''}{x'} - 3\left(\frac{x''}{x'}\right)^2 + 3\,r\,(x')^2 = 0 \qquad (o)$$

an equation connecting x and t directly.

8. It remains to be proved that (o) is reducible to a linear differential equation of the second order. To this end, let

$$x' = p \qquad (p)$$

and consequently

$$x'' = p' = p' \,.\, p \qquad (q)$$

and also

$$x''' = p'' = p'' \,.\, p^2 + (p')^2 \,.\, p \qquad (r)$$

then (o) becomes, after substitutions,

$$2\,p\,p'' + 2\,(p')^2 - 3\,(p')^2 + 3\,r\,p^2 = 0,$$

and, on reduction,

$$2pp'' - (p')^2 + 3rp^2 = 0 \quad - \quad - \quad - \quad - \quad (s)$$

Now divide this result by p^2 and we have

$$2\frac{p''}{p} - \left(\frac{p'}{p}\right)^2 + 3r = 0 \quad - \quad - \quad - \quad (t)$$

or

$$2\frac{d}{dx}\left(\frac{p'}{p}\right) + \left(\frac{p'}{p}\right)^2 + 3r = 0 \quad - \quad - \quad (u)$$

and if we make

$$\frac{p'}{p} = 2v \quad - \quad - \quad - \quad - \quad - \quad (v)$$

then (u) becomes, on dividing by 4

$$\frac{dv}{dx} + v^2 + \frac{3}{4}r = 0 \quad - \quad - \quad - \quad (w)$$

whence also

$$e^{\int v\,dx}\left(\frac{dv}{dx} + v^2 + \frac{3}{4}r\right) = 0 \quad - \quad - \quad (x)$$

or, making $e^{\int v\,dx} = w$, - - - - - - (y)

$$\frac{d^2w}{dx^2} + \frac{3}{4}rw = 0 \quad - \quad - \quad - \quad (z)$$

a linear differential equation of the second order.

9. From (y) we deduce

$$v = \frac{1}{w} \cdot \frac{dw}{dx} \quad - \quad - \quad - \quad (aa)$$

and, combining this with (v) we find, on integration, &c.,

$$p = C_2 w^2 \quad - \quad - \quad - \quad - \quad (ab)$$

whence, by (p),

$$\frac{dt}{dx} = \frac{1}{p} = \frac{1}{C_2 w^2} \quad - \quad - \quad - \quad (ac)$$

or

$$t = \frac{1}{C_2}\int \frac{dx}{w^2} + C_3 \quad - \quad - \quad - \quad (ad)$$

We find also, combining (i) and (p) and (ab) that

$$u = C\,C_2 w^2 \quad - \quad - \quad - \quad - \quad (ae)$$

which two constants are of course equivalent to one only.

10. The transformation thus indicated for the third order is possible for equations of any order. For starting with the equation, deprived of its second term and in which n is greater than 3,

$$\frac{d^n y}{d x^n} + r \frac{d^{n-2} y}{d x^{n-2}} + \text{\&c.} = 0 \qquad - \quad - \quad (\alpha)$$

we obtain by means of Mr. S. S. Greatheed's general formulæ for the change of the independent variable (in the *Cambridge Mathematical Journal*, vol. i. pp. 236—8) the following results, true for all values of n:

$$\frac{d^n y}{d x^n} = q_1 \frac{d y}{d t} + q_2 \frac{d^2 y}{d t^2} + \ldots q_n \frac{d^n y}{d t^n}, \qquad - \quad (\beta)$$

where

$$q_p = M \left\{ \left(\frac{d^\lambda t}{d x^\lambda}\right)^\alpha \left(\frac{d^\mu y}{d x^\mu}\right)^\beta \left(\frac{d^\nu y}{d x^\nu}\right)^\gamma \right\} \qquad - \quad (\gamma)$$

M meaning a multiple of the included quantities, and α, β, γ, &c, and λ, μ, ν, &c., being subject to the two relations

$$\alpha + \beta + \gamma + \ldots = p \qquad - \quad - \quad - \quad (\delta)$$

$$\alpha\lambda + \beta\mu + \gamma\nu + \ldots = n \qquad - \quad - \quad - \quad (\epsilon)$$

11. We are only concerned with the first three terms of the transformed equation. For the first put $p = n$ and subtract (δ) from (ϵ). The result is

$$\alpha(\lambda - 1) + \beta(\mu - 1) + \gamma(\nu - 1) + \ldots = 0 \qquad - \quad (\zeta)$$

of which, since α, &c., and λ, &c., are to be integers, the only available solutions are

$$\lambda = 1, \ \mu = 1, \ \nu = 1, \ \ldots \ \alpha = 1, \ \beta = 1, \ \gamma = 1, \ \ldots$$

consequently

$$q_n = M \left\{ \left(\frac{d t}{d x}\right)^n \right\} \qquad - \quad - \quad - \quad (\eta)$$

Next, let $p = n - 1$, then subtracting as before

$$\alpha(\lambda - 1) + \beta(\mu - 1) + \gamma(\nu - 1) + \ldots = 1 \qquad - \quad (\theta)$$

the only available solutions of which are of the form

$$\lambda = 2, \ \mu = 1, \ \nu = 1, \ \ldots \ \alpha = 1, \ \beta = 1, \ \gamma = 1, \ \ldots$$

so that

$$q_{n-1} = M \left\{ \frac{d^2 t}{d x^2} \left(\frac{d t}{d x}\right)^{n-1} \right\} \qquad - \quad - \quad (\iota)$$

Again, when $p=n-2$, we have, subtracting as before, the condition

$$\alpha(\lambda-1)+\beta(\mu-1)+\gamma(\nu-1)+\ldots=2 \qquad - \quad (\kappa)$$

of which the only available solutions are of the forms

$$\lambda=3,\ \mu=1,\ \nu=1,\ \rho=1,\ \ldots\ \alpha=1,\ \beta=1,\ \ldots$$

and

$$\lambda=2,\ \mu=2,\ \nu=1,\ \rho=1,\ \ldots\ \alpha=1,\ \beta=1,\ \ldots$$

Hence

$$q_{n-2}=M\left\{\frac{d^3t}{dx^3}\left(\frac{dt}{dx}\right)^{n-1}\right\}+M_2\left\{\left(\frac{d^2t}{dx^2}\right)^2\left(\frac{dt}{dx}\right)^{n-2}\right\} \quad (\lambda)$$

and we see that the conditions for the annihilation of the second and third terms of an equation of any order will not essentially differ from those for equations of the third order already discussed.

The forms of the conditions also show that the simultaneous destruction of the second and rth terms of a linear differential equation of any order may be made to depend upon the solution of equations of the first and rth order: and thus far the analogy between algebra and the calculus holds—to a certain extent at least. I have not ascertained whether the resulting equation in the case of r being greater than 2 can be made linear.

12. The sinister of a linear (or simple) algebraical equation whose dexter is zero may be reduced to a single term by an easy transformation, and the sinister of a linear differential equation of the first order whose dexter is zero possesses the analogous property that, being multiplied into an appropriate factor, it may be reduced to the form of a perfect differential coefficient. We will now inquire whether the analogy between algebra and the differential calculus holds in the case of the next degree and order. And in so doing it will no longer be necessary for me to adhere to the abridged notation so conducive to the perspicuity of foregoing results and to the manageability of the formulæ involved in them. I proceed to attempt the annihilation of the last two terms of a linear differential equation of the second order, and for a reason given in Article 2 I shall start from the linear differential equation of the second order

$$\frac{d^2y}{dx^2}+ry=0 \qquad - \quad - \quad - \quad (1)$$

the transformation of which has no analogue in the theory of algebraical equations, inasmuch as a quadratic cannot be deprived of its second and last terms simultaneously. We shall thus see whether there is any instance in which the analogy fails to hold between algebra and the differential calculus.

13. To effect the proposed transformation of (1) I change the independent variable from x to t and, at the same time, the dependent variable from y to Y, the relation

$$y = u\,Y \qquad \text{- - - - (2)}$$

subsisting between y and Y. The result of these changes is to transform (1) into

$$\frac{d^2(u\,Y)}{d\,t^2}\cdot\left(\frac{d\,t}{d\,x}\right)^2 - \frac{d\,(u\,Y)}{d\,t}\cdot\left(\frac{d\,t}{d\,x}\right)^3\cdot\frac{d^2 x}{d\,t^2} + r\,(u\,Y) = 0 \quad (3)$$

or

$$\frac{d^2(u\,Y)}{d\,t^2} - \frac{d\,t}{d\,x}\cdot\frac{d^2 x}{d\,t^2}\cdot\frac{d\,(u\,Y)}{d\,t} + r\left(\frac{d\,x}{d\,t}\right)^2(u\,Y)$$

14. Developing, we have

$$\frac{d^2(u\,Y)}{d\,t^2} = u\frac{d^2 Y}{d\,t^2} + 2\,\frac{d\,u}{d\,t}\cdot\frac{d\,Y}{d\,t} + \frac{d^2 u}{d\,t^2}\,Y$$

$$-\frac{d\,t}{dx}\cdot\frac{d^2x}{d\,t^2}\cdot\frac{d(uY)}{d\,t} = \quad -u\frac{d\,t}{dx}\cdot\frac{d^2x}{d\,t^2}\cdot\frac{d\,Y}{dt} - \frac{du}{d\,t}\cdot\frac{d\,t}{dx}\cdot\frac{d^2x}{d\,t^2}\,Y$$

$$r\left(\frac{d\,x}{d\,t}\right)^2(u\,Y) = \quad r\,u\left(\frac{d\,x}{d\,t}\right)^2 Y.$$

Hence, (4) may, after division by u, be replaced by

$$\left.\begin{aligned}&\frac{d^2 Y}{d\,t^2} + \left(\frac{2}{u}\cdot\frac{d\,u}{d\,t} - \frac{d\,t}{d\,x}\cdot\frac{d^2 x}{d\,t^2}\right)\frac{d\,Y}{d\,t}\\ &+\left(\frac{1}{u}\cdot\frac{d^2 u}{d\,t^2} - \frac{1}{u}\cdot\frac{d\,u}{d\,t}\cdot\frac{d\,t}{d\,x}\cdot\frac{d^2 x}{d\,t^2} + r\left\{\frac{d\,x}{dt}\right\}^2\right)Y = 0\end{aligned}\right\}\quad (5)$$

15. Hence the conditions of the transformation are

$$\frac{2}{u}\cdot\frac{d\,u}{d\,t} - \frac{d\,t}{d\,x}\cdot\frac{d^2 x}{d\,t^2} = 0 \qquad \text{- - - (6)}$$

and

$$\frac{1}{u}\cdot\frac{d^2 u}{d\,t^2} - \frac{1}{u}\cdot\frac{d\,u}{d\,t}\cdot\frac{d\,t}{d\,x}\cdot\frac{d^2 x}{d\,t^2} + r\left(\frac{d\,x}{d\,t}\right)^2 = 0 \qquad \text{- (7)}$$

which equations I proceed to solve.

16. From (6), that is to say from

$$\frac{2}{u} \cdot \frac{d\,u}{d\,t} = \frac{\dfrac{d^2 x}{d\,t^2}}{\dfrac{d\,x}{d\,t}}, \quad - \quad - \qquad (8)$$

we deduce by integration

$$\log.\,(u^2) = \log.\left\{ C\left(\frac{d\,x}{d\,t}\right) \right\}$$

or

$$u^2 = C \cdot \frac{d\,x}{d\,t} \quad - \quad - \quad - \quad - \quad (9)$$

and by differentiation

$$\frac{2}{u} \cdot \frac{d^2 u}{d\,t^2} - \frac{2}{u^2} \cdot \left(\frac{d\,u}{d\,t}\right)^2 = \frac{\dfrac{d^3 x}{d\,t^3}}{\dfrac{d\,x}{d\,t}} - \left\{ \frac{\dfrac{d^2 x}{d\,t^2}}{\dfrac{d\,x}{d\,t}} \right\}^2 \qquad (10)$$

and (8) enables us to write (7) in the form

$$\frac{1}{u} \cdot \frac{d^2 u}{d\,t^2} - 2\left(\frac{1}{u} \cdot \frac{d\,u}{d\,t}\right)^2 + r\left(\frac{d\,x}{d\,t}\right)^2 = 0 \quad - (11)$$

from which if we eliminate u by means of (8) and (10) we obtain

$$\frac{1}{2} \cdot \frac{\dfrac{d^3 x}{d\,t^3}}{\dfrac{d\,x}{d\,t}} - \frac{3}{4} \cdot \left(\frac{\dfrac{d^2 x}{d\,t^2}}{\dfrac{d\,x}{d\,t}} \right)^2 + r\left(\frac{d\,x}{d\,t}\right)^2 = 0 \quad - (12)$$

17. Now assume

$$\frac{d\,x}{d\,t} = p \quad - \quad - \quad - \quad - (13)$$

whence

$$\frac{d^2 x}{d\,t^2} = \frac{d\,p}{d\,t} = \frac{d\,p}{d\,x} \cdot p, \text{ and } \frac{d^3 x}{d\,t^3} = \frac{d^2 p}{d\,x^2} \cdot p^2 + \left(\frac{d\,p}{d\,x}\right)^2 p \quad (14, 15)$$

and eliminate t from (12). That equation becomes

$$\frac{1}{2} \cdot p\,\frac{d^2 p}{d\,x^2} + \frac{1}{2}\left(\frac{d\,p}{d\,x}\right)^2 - \frac{3}{4}\left(\frac{d\,p}{d\,x}\right)^2 + r\,p^2 = 0 \quad - (16)$$

or, multiplying (16) into 4 and reducing

$$2\,p\,\frac{d^2 p}{d\,x^2} - \left(\frac{d\,p}{d\,x}\right)^2 + 4\,r\,p^2 = 0 \quad - \quad - (17)$$

18. This equation is of the second order. It remains to be shown that its solution depends upon that of a linear equation of the same order. To this end divide it by p^2 and it becomes

$$\frac{2}{p}\cdot\frac{d^2 p}{d\,x^2} - \frac{1}{p^2}\left(\frac{d\,p}{d\,x}\right)^2 + 4\,r = 0 \quad - \quad - (18)$$

which is equivalent to

$$2\,\frac{d}{d\,x}\left\{\frac{1}{p}\cdot\frac{d\,p}{d\,x}\right\} + \left(\frac{1}{p}\cdot\frac{d\,p}{d\,x}\right)^2 + 4\,r = 0 \quad - (19)$$

19. Now if we put

$$\frac{1}{p}\cdot\frac{d\,p}{d\,x} = a\,v \quad - \quad - \quad - \quad - (20)$$

then (19) becomes

$$2\,a\,\frac{d\,v}{d\,x} + a^2\,v^2 + 4\,r = 0 \quad - \quad - \quad - (21)$$

which if $a = 2$ reduces to

$$\frac{d\,v}{d\,x} + v^2 + r = 0 \quad - \quad - \quad - (22)$$

and so to

$$e^{\int v\,d\,x}\left(\frac{d\,v}{d\,x} + v^2 + r\right) = 0$$

or (making

$$e^{\int v\,d\,x} = w)$$

to

$$\frac{d^2\,w}{d\,x^2} + r\,w = 0 \quad - \quad - \quad - \quad - (23)$$

20. Now, retracing our steps, we see that (1) and (23) are identical in form, and also that (compare article 9)

$$p = \frac{d\,x}{d\,t} = C_2\,w^2 \quad - \quad - \quad - \quad - (24)$$

whence, by equation (9),

$$u^2 = C\frac{dx}{dt} = C\,C_2\,w^2 \quad - \quad - \quad - \quad - \;(25)$$

$$\text{and } \therefore\; u = M\,w \quad - \quad - \quad - \quad - \quad - \quad - \;(26)$$

M being a constant. The analogy between the algebraical and differential theories holds then thus far: the supposition that u or, which is the same thing, w is known, is the same as supposing that the solution of (1) the given equation, is known, and consequently to effect the transformation we have to encounter all the difficulties of solving the original equation. The analogy fails thus far: if we can solve (1) we can annihilate its final term. This failure of analogy seems analogous to another failure of analogy between quadratics and linear differential equations of the second order: a quadratic may have equal roots, but there are two arbitrary constants in the complete integral of every linear differential equation of the second order.

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§ 2. *On Co-resolvents.*

21. The Theory of Co-resolvents originated in my "Sketch of a Theory of Transcendental Roots," published in the *Philosophical Magazine* for August, 1860. The subject of that paper has since been pursued as well by myself as by Mr. Harley, Professor Cayley, Mr. Russell, Mr. Rawson, Mr. Spottiswoode, and the lamented Boole. When two or more conditions involving a quantity are simultaneously satisfied by the same value of the quantity, those conditions, and indeed that by which the value is determined, may be termed "Co-resolvents," and one co-resolvent may sometimes not inaptly be termed a "resolvent" of another. The co-resolvents may be 'algebraical resolvents,' or 'differential resolvents,' or 'functional resolvents.' Thus the three equations

$$y^3 - 3\,y + 2\,x = 0 \quad - \quad - \quad - \;(\text{i})$$

$$3^2\,(1 - x^2)\,\frac{d^2 y}{d\,x^2} - 3^2\,x\,\frac{d\,y}{d\,x} + y = 0 \quad - \;(\text{ii})$$

$$\frac{d\,\phi\,(x)}{d\,x} + \frac{\phi\,(\sqrt{1 - x^2})}{3\,\sqrt{1 - x^2}} = 0\,, \quad - \quad - \;(\text{iii})$$

in the first two of which y is to be taken to represent $\phi\ (x)$, are co-resolvents. The first is the algebraical resolvent, the second is the differential resolvent, and the third is a functional resolvent. If the variable x be not greater than positive unity, or less than negative unity, the relation

$$\phi\ (x) = 2 \sin. \left(\frac{2\,\pi + \sin.^{-1} x}{3}\right) \qquad \text{(iv)}$$

is a solution, and the only common solution, of the above system of co-resolvents, while, on the other hand, all the roots of (i) are solutions of (ii). The theory of co-resolvents not only throws light on those of algebraical and of differential equations, but it enables us to make the solution of whole classes of functional equations depend upon that of algebraical or of differential equations. The well-known application of the theory of algebraical equations to the solution of linear differential equations with constant coefficients, the analogies between the theories of algebraical and differential equations pointed out by Libri and Lioüville and the communication between the two theories established by Abel, when he explored the track upon which Euler had entered, may give interest to the new communication between those theories opened by the method of co-resolvents.

22. Before proceeding to that development of the theory which it is the object of this section to explain, I ought to say that its present advanced state is, in no small measure, attributable to Mr. Harley. That eminent mathematician, by his earlier inquiries into the forms of the differential resolvents of certain trinomial algebraical equations, obtained results which not only excited attention at the time, but which have also, to a great extent, determined the current of subsequent research. Traces of Mr. Harley's investigations appear in almost every paper that has since been published. His approach to the following theorem was simultaneous with my own.

23. If u represent the mth power of any root of the algebraical equation

$$y^n - x\,y^r - 1 = 0 \qquad \text{(v)}$$

then u, considered as a function of x, satisfies the linear differential equation

$$[D]^n u - \left[\frac{m + rD}{n} - 1\right]^r \left[\frac{m - (n-r)\,D}{n} + n - r\right]^{n-r} x^n u = 0 \qquad \text{(vi)}$$

in which D is an operative symbol defined by

$$D = x \frac{d}{dx} \quad - \quad - \quad - \quad - \text{(vii)}$$

and the notation

$$[a]^b = a\,(a-1)\,(a-2)\,..\,(a-b+1) \; - \qquad \text{(viii)}$$

is adopted. This theorem is an extension of Boole's proposition at pages 734 and 735 of the *Philosophical Transactions* for 1864. If we make

$$r = n - 1\,, \; x = e^{\theta}$$

and therefore regard D as representing the operation $\frac{d}{d\theta}$, then (vi) will become

$$[D]^n u + \left[\frac{n-1}{n} D + \frac{m}{n} - 1\right]^{n-1} \left(\frac{D}{n} - \frac{m}{n} - 1\right) e^{n\theta} u = 0 \quad \text{(ix)}$$

and so coincide with the equation given by Boole at the pages cited.

24. As a second verification of this theorem let $r = 1$, then (vi) becomes

$$[D]^n u - \frac{1}{n}(D - n + m)\left[\frac{m}{n} - \frac{n-1}{n} D + n - 1\right]^{n-1} x^n u = 0 \quad \text{(x)}$$

which, when $m = 1$, becomes

$$[D]^n y - \frac{1}{n}(D - n + 1)\left[\frac{1}{n} - \frac{n-1}{n} D + n - 1\right]^{n-1} x^n y = 0 \quad \text{(xi)}$$

or, which is the same thing,

$$[D]^n y + \frac{1}{n}\left[\frac{n-1}{n} D - \frac{1}{n} - 1\right]^{n-1} (D - n + 1)(-x)^n y = 0 \quad \text{(xii)}$$

which is the n-ary differential resolvent of

$$y^n - x\,y - 1 = 0 \; - \quad - \quad - \qquad \text{(xiii)}$$

Now if in (xii) we replace x by $-x$, and afterwards, as before, substitute e^{θ} for x, then (xii) becomes

$$[D]^n y + \frac{1}{n}\left[\frac{n-1}{n} D - \frac{1}{n} - 1\right]^{n-1} (D - n + 1) e^{n\theta} y = 0 \quad \text{(xiv)}$$

which is the n-ary differential resolvent of

$$y^n + x\,y - 1 = 0 \qquad - \quad - \qquad \text{(xv)}$$

This result coincides ultimately with Boole's. For if in (xv) we replace y by y^{-1} we have an equation employed by Boole (*ibid.* p. 736) and if in (xiv) we replace y by u we have its n-ary resolvent as given by him (*ibid.* p. 737).

25. If we make

$$G = \frac{r}{n} D + \frac{m}{n} - 1 \qquad \text{(xvi)}$$

and

$$H = \frac{m - (n - r) D}{n} + n - r \qquad \text{(xvii)}$$

the general theorem may conveniently be expressed as follows :—The n-ary, or Boolian differential resolvent of (v) is

$$[D]^n u - [G]^r [H]^{n-r} x^n u = 0 \qquad \text{(xviii)}$$

26. I communicated the generalization of Boole's proposition to Mr. Harley by the last October mail, together with a verification—both verifications, I believe. By the last December mail I received a letter from Mr. Harley, dated Oct. 17, 1865, which therefore crossed my letter to him, and in which he makes a very near approach indeed to the true generalization : so near, indeed, that, inasmuch as the oversight into which he has fallen could not long have escaped his notice, the generalization may be regarded as having been independently made by me in Queensland, Australia, and by him in England. By the last (February) mail I received from Mr. Harley a letter dated Dec. 18, 1865, in which he acknowledges my letter of the 18th Oct. and in substance says that the true generalization is

$$[D]^n u - (-)^{n-r} [G]^r [H]^{n-r} x^{n-r} u = 0 \qquad \text{(xix)}$$

where

$$H = \frac{n - r}{n} D - \frac{m}{n} - 1 \qquad \text{(xx)}$$

Now, inasmuch as

$$\left[\frac{m-(n-r)D}{n} + n - r\right]^{n-r} = \left[\frac{n-r}{n} D - \frac{m}{n} - 1\right]^{n-r} (-1)^{n-r} \quad \text{(xxi)}$$

the only real discrepancy between our results consists in the different indices of x. Mr. Harley's factor 'x^{n-r}' seems to me erroneous, and neither to follow from the law of derivation of a differential equation from a series, nor to agree either with Boole's results or with obvious properties of the n-ary resolvent of the equation in y or y^m. If the general n-ary resolvent of (v) be denoted by

$$f(m, x) = 0 \quad - \quad - \quad - \quad - \quad \text{(xxii)}$$

that of

$$\left(\frac{1}{y}\right)^n + x\left(\frac{1}{y}\right)^{n-r} - 1 = 0 \quad - \quad \text{(xxiii)}$$

or, what is the same thing, of

$$(y^{-1})^n + x\,(y^{-1})^{n-r} - 1 = 0 \quad - \quad - \quad \text{(xxiv)}$$

will still be (xxii), for (xxiii) is but the result of the division of (v) by y^n. Hence if in (xxiv) we replace y^{-1} by z and x by $-x$, we must replace m by $-m$ and x by $-x$ in (xxii) in order to obtain the n-ary resolvent of

$$z^n - x\,z^{n-r} - 1 = 0 \quad - \quad - \quad - \quad \text{(xxv)}$$

that resolvent is, consequently,

$$f(-m, -x) = 0 \quad - \quad - \quad - \quad \text{(xxvi)}$$

in other words when, in (v) we change r into $n-r$ the index of x in the resolvent remains unaltered.

27. The demonstration of the generalized theorem is as follows. Let

$$y^n - x\,y^a - 1 = 0 \quad - \quad - \quad \text{(xxvii)}$$

then by Lagrange's theorem, employed precisely as Boole has done (*ibid.* p. 735), we find that u or y^m may be expanded in a series of the form

$$u_0 + u_1 x + u_2 x^2 + \&c.\ ad.\ inf.,$$

in which

$$u_r = \frac{m\left[\dfrac{m+a\,r}{n} - 1\right]^{r-1} \times (1)^{\frac{m-r}{n}}}{n\,[r]^r} \quad - \quad \text{(xxviii)}$$

and consequently

$$[r]^n\,u_r = \frac{m\left[\dfrac{m+a\,r}{n} - 1\right]^{r-1} \cdot (1)^{\frac{m-r}{n}}}{n\,[r-n]^{r-n}} \quad - \quad \text{(xxix)}$$

28. Changing r into $r - n$ we have, from (xxviii),

$$u_{r-n} = \frac{m\left[\dfrac{m+a\,r}{n} - a - 1\right]^{r-n-1} \cdot (1)^{\frac{m-r}{n}}}{n\,[r-n]^{r-n}} \qquad \text{(xxx)}$$

Hence, dividing (xxix) by (xxx),

$$\frac{[r]^n u_r}{u_{r-n}} = \frac{\left[\dfrac{m+a\,r}{n} - 1\right]^{r-1}}{\left[\dfrac{m+a\,r}{n} - a - 1\right]^{r-n-1}} \qquad \text{(xxxi)}$$

But

$$\left[\frac{m+a\,r}{n} - a - 1\right]^{r-n-1} = \frac{\left[\dfrac{m+a\,r}{n} - 1\right]^{r-n-1+a}}{\left[\dfrac{m+a\,r}{n} - 1\right]^{a}}$$

$$= \frac{\left[\dfrac{m+a\,r}{n} - 1\right]^{r-1}}{\left[\dfrac{m+a\,r}{n} - 1\right]^{a}\left[\dfrac{m+a\,r}{n} - r + n - a\right]^{n-a}} \qquad \text{(xxxi)}$$

consequently

$$\frac{[r]^n u_r}{u_{r-n}} = \left[\frac{m+a\,r}{n} - 1\right]^{a}\left[\frac{m+a\,r}{n} - r + n - a\right]^{n-a}$$

or, making two obvious changes in the form of this result

$$[r]^n u_r - \left[\frac{m+a r}{n} - 1\right]^{a}\left[\frac{m-(n-a)\,r}{n} + n - a\right]^{n-a} u_{r-n} = 0 \qquad \text{(xxxii)}$$

This equation corresponds to Boole's equation (6) (*ibid.* p. 736) which requires correction in two places and should stand thus:

$$[r]^n u_r + \left[\frac{m+(n-1)r}{n} - 1\right]^{n-1}\left(\frac{r}{n} - \frac{m}{n} - 1\right) u_{r-n} = 0 \qquad (6)$$

29. Multiply both sides of (xxxii) into x^r and it becomes

$$[r]^n u_r \,.\, x^r - \left[\frac{m+a r}{n} - 1\right]^{a}\left[\frac{m-(n-a)\,r}{n} + n - a\right]^{n-a} .\, x^n \,.\, u_{r-n} .\, x^{r-n} = 0 \qquad \text{(xxxiii)}$$

or (since

$$x \frac{d}{d\,x}(x^k) = k\,\Big)$$

$$[D]^n u_r \, . \, x^r - \left[\frac{m + aD}{n} - 1\right]^a \left[\frac{m-(n-a)\,D}{n} + n - a\right]^{n-a} x^n \, . \, u_{r-n} \, . \, x^{r-n} = 0$$

(xxxiv)

Next, in (xxxiv), give to r all successive values from zero to infinity and add the results : we find, since

$$u_0 + u_1 x + u_2 x^2 + \text{\&c. } ad.\ inf. = u,$$

$$[D]^n u - \left[\frac{m + aD}{n} - 1\right]^a \left[\frac{m-(n-a)\,D}{n} + n - a\right]^{n-a} x^n u = 0 \quad \text{(xxxv)}$$

Hence if in (xxvii) we replace a by r and, consequently, make the same change in (xxxv) that last equation becomes

$$[D]^n u - \left[\frac{m + rD}{n} - 1\right]^r \left[\frac{m-(n-r)\,D}{n} + n - r\right]^{n-r} x^n u = 0 \quad \text{- (vi)}$$

and thus the theorem announced at the commencement of Article 23 is established. The constant r which replaces the constant a will not of course be confounded with the variable integer r used in the investigations immediately preceding.

Brisbane, Queensland, Australia,
March 16, 1866.

Abstract of the Addendum to CHIEF JUSTICE COCKLE'S *Paper "On Differential Equations," &c., communicated to* SIR REDMOMD BARRY.

In an Addendum to the above paper, dated April 9th-10th, 1866, Chief Justice Cockle gives a co-resolvent of a third form of trinomial algebraic equation undiscussed by Mr. Harley and the lamented Boole. After obtaining this last co-resolvent by means of Laplace's theorem, the Chief Justice gives a general theorem for the development of functions of two independent quantities. Lagrange's theorem and Laplace's theorem are particular cases of this general theorem.

Addendum to the foregoing Paper.

Under date Oakwal, near Brisbane, April 9, 1866, Chief Justice Cockle communicated to Sir Redmond Barry the following additions to his Paper " On Differential Equations and on Co-resolvents."

I. We may readily deduce a differential co-resolvent of

$$x\,y^n - y^a + z = 0\,, \qquad \text{(a)}$$

from which we obtain successively

$$y^a = z + x\,y^n \qquad \text{(b)}$$

$$y = (z + x\,y^n)^{\frac{1}{a}} \qquad \text{(c)}$$

$$y^m = (z + x\,y^n)^{\frac{m}{a}} \qquad \text{(d)}$$

Hence, by Laplace's theorem,

$$y^m = z^{\frac{m}{a}} + \left\{ z^{\frac{n}{a}} \left(\frac{m}{a} z^{\frac{m}{a}-1} \right) \right\} \frac{x}{1} + \frac{d}{dz} \left\{ z^{\frac{2n}{a}} \left(\frac{m}{a} z^{\frac{m}{a}-1} \right) \right\} \frac{x^2}{1\,.\,2} + \&\text{c.} \ \text{(e)}$$

II. Now since the general term of the series for y^m is

$$\frac{m}{a} \cdot \frac{d^{r-1}}{d\,z^{r-1}} \left\{ z^{\frac{rn}{a}} \,.\, z^{\frac{m}{a}-1} \right\} \frac{x^r}{1.2..r}$$

or

$$\frac{m}{a} \cdot \frac{d^{r-1}}{d\,z^{r-1}} \left\{ z^{\frac{rn}{a}+\frac{m}{a}-1} \right\} \frac{x^r}{[r]^r}$$

or, if we make $z = 1$ after the differentiations,

$$\frac{m}{a} \left[\frac{r\,n}{a} + \frac{m}{a} - 1 \right]^{r-1} \frac{x^r}{[r]^r}$$

we may put

$$[r]^r\, u_r = \frac{m}{a} \left[\frac{r\,n}{a} + \frac{m}{a} - 1 \right]^{r-1} \qquad \text{(f)}$$

and, consequently,

$$[r - a]\, u_{r-a} = \frac{m}{a} \left[\frac{(r-a)\,n}{a} + \frac{m}{a} - 1 \right]^{r-a-1} \qquad \text{(g)}$$

III. It follows that

$$[r]^a u_r = \frac{\left[\frac{rn}{a} + \frac{m}{a} - 1\right]^{r-1}}{\left[\frac{rn}{a} + \frac{m}{a} - 1 - n\right]^{r-a-1}} u_{r-a} \quad - \quad - \quad \text{(h)}$$

But

$$\left[\frac{rn}{a} + \frac{m}{a} - (1+n)\right]^{r-a-1} = \frac{\left[\frac{rn}{a} + \frac{m}{a} - 1\right]^{r-a-1+n}}{\left[\frac{rn}{a} + \frac{m}{a} - 1\right]^n}$$

$$= \frac{\left[\frac{rn}{a} + \frac{m}{a} - 1\right]^{r-1} \left[\frac{rn}{a} + \frac{m}{a} - r\right]^{n-a}}{\left[\frac{rn}{a} + \frac{m}{a} - 1\right]^n} \quad - \quad \text{(i)}$$

and (h) becomes

$$[r]^a u_r = \frac{\left[\frac{rn}{a} + \frac{m}{a} - 1\right]^n}{\left[\frac{rn}{a} + \frac{m}{a} - r\right]^{n-a}} \cdot u_{r-a} \quad - \quad - \quad \text{(j)}$$

IV. Proceeding as in the former part of this paper we, slightly changing the form of (j), infer that, since

$$[r]^a \left[\frac{(n-a)r}{a} + \frac{m}{a}\right]^{n-a} u_r - \left[\frac{rn}{a} + \frac{m}{a} - 1\right]^n u_{r-a} = 0 \quad \text{(k)}$$

consequently that, as in a former case, the co-resolvent will be

$$[D]^a \left[\frac{(n-a)D}{a} + \frac{m}{a}\right]^{n-a} u - \left[\frac{nD}{a} + \frac{m}{a} - 1\right]^n x^a u = 0 \quad \text{(l)}$$

in which last equation u represents y^m. This (l) is the co-resolvent of (a) for the case of $z = 1$. We may obtain it for any value of z by multiplying the last term of (l) into z^{n-a}, for the exponent of z will be

$$\frac{rn}{a} + \frac{m}{a} - r - \left(\frac{(r-a)n}{a} + \frac{m}{a} - r + a\right) = n - a$$

We shall thus give to the co-resolvent an extension equivalent to that which Mr. Harley has sought to give to another result. I find that Mr. Harley's extension of the theorem spoken of in the former part of this memoir is printed at p. 199 of Boole's (posthumous) Supplementary Volume, which reached me by the last (March) mail. The extension as there printed is erroneous to an extent which I have already pointed out.

V. In general, let

$$u = f(y) \qquad \text{(m)}$$

$$y = \psi(w) \qquad \text{(n)}$$

$$w = (az + bx)F(y) + (ax + bz)\pi(y) + \chi(y) \qquad \text{(o)}$$

in which x and z are independent, and F, π, and χ are functional symbols, and let it be required to expand u in a series of ascending powers of x.

VI. Put

$$(az + bx)\frac{dF(y)}{dy} + (ax + bz)\frac{d\pi(y)}{d(y)} + \frac{d\chi(y)}{d(y)} = H \qquad \text{(p)}$$

and we have

$$\frac{dy}{dx} = \frac{d\psi(w)}{dw} \cdot \frac{dw}{dx} = \frac{d\psi(w)}{dw} \cdot \left\{H\frac{dy}{dx} + bF(y) + a\pi(y)\right\} \qquad \text{(q)}$$

In like manner

$$\frac{dy}{dz} = \frac{d\chi(w)}{dw}\left\{H\frac{dy}{dz} + aF(y) + b\pi(y)\right\} \qquad \text{(r)}$$

and therefore, putting (q) and (r) under the forms

$$\frac{dy}{dx}\left\{1 - H\frac{d\psi(w)}{dw}\right\} = \frac{d\psi(w)}{dw}\left\{bF(y)\,a\pi(y)\right\} \qquad \text{(s)}$$

$$\frac{dy}{dz}\left\{1 - H\frac{d\psi(w)}{dw}\right\} = \frac{d\psi(w)}{dw}\left\{aF(y) + b\pi(y)\right\} \qquad \text{(t)}$$

and dividing (s) by (t) we have, after a slight reduction,

$$\frac{dy}{dx} = \left\{\frac{bF(y) + a\pi(y)}{aF(y) + b\pi(y)}\right\}\frac{dy}{dz} \qquad \text{(u)}$$

VII. For simplicity, divide both the numerator and denominator of the bracketed factor of the dexter of (u) by $F(y)$, then (u) becomes

$$\frac{dy}{dx} = \left\{\frac{b + a\phi(y)}{a + b\phi(y)}\right\}\frac{dy}{dz} = \mu(y)\frac{dy}{dz} \qquad \text{(v)}$$

where

$$\phi(y) = \frac{\pi(y)}{F(y)} \quad - \quad - \quad - \quad - \quad - \quad - \quad - \text{(w)}$$

and μ is a new functional symbol introduced for convenience.

VIII. Now

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \mu(y) \frac{du}{dy} \cdot \frac{dy}{dz} = \mu(y) \frac{du}{dz} \quad - \quad - \text{(x)}$$

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left\{\mu(y)\frac{du}{dz}\right\} = \frac{d}{dz}\left\{\mu(y)\frac{du}{dx}\right\} = \frac{d}{dz}\left\{\left(\mu(y)\right)^2\frac{du}{dz}\right\} \quad \text{(y)}$$

$$\frac{d^3u}{dx^3} = \frac{d}{dx} \cdot \frac{d}{dz}\left\{\left(\mu(y)\right)^2\frac{du}{dz}\right\} = \frac{d^2}{dz^2}\left\{\left(\mu(y)\right)^3\frac{du}{dz}\right\} - \text{(z)}$$

and so on. For we know that

$$\frac{d}{dx}\left\{V\frac{dv}{dz}\right\} = \frac{d}{dz}\left\{V\frac{dv}{dx}\right\}$$

where v is any function of the independent quantities x and z, and V any function of v.

IX. Next, denoting by a suffixed zero the value which a function takes when x vanishes (for instance, denoting by K_0 the value of K when $x = 0$) we have, by Maclaurin's theorem,

$$u = u_0 + \left(\frac{du}{dx}\right)_0 \cdot \frac{x}{1} + \left(\frac{d^2u}{dx^2}\right)_0 \cdot \frac{x^2}{1.2} + \&c. \quad - \quad \text{(aa)}$$

Hence, by what has preceded,

$$u = u_0 + \mu(y_0) \cdot \frac{du_0}{dz} \cdot \frac{x}{1} + \frac{d}{dz}\left\{\left(\mu(y_0)\right)^2 \cdot \frac{du_0}{dz}\right\} \cdot \frac{x^2}{1.2} +$$

a series whose general term is

$$\frac{d^{r-1}}{dz^{r-1}}\left\{\left(\mu(y_0)\right)^r\frac{du_0}{dz}\right\} \cdot \frac{x^r}{[r]^r} \quad - \quad - \text{(ab)}$$

X. We have now to express u_0 or $f(y_0)$ as a function of z. For this purpose making $x = 0$ in (o) we deduce by means of (m), (n) and (o)

$$u_0 = f(y_0) = f\psi(w_0) = f\psi\{azF(y_0) + bz\pi(y_0) + \chi(y_0)\} \quad \text{(ac)}$$

whence

$$\psi^{-1}(y_0) - \chi(y_0) = z\left(aF(y_0) + b\pi(y_0)\right) \quad \text{(ad)}$$